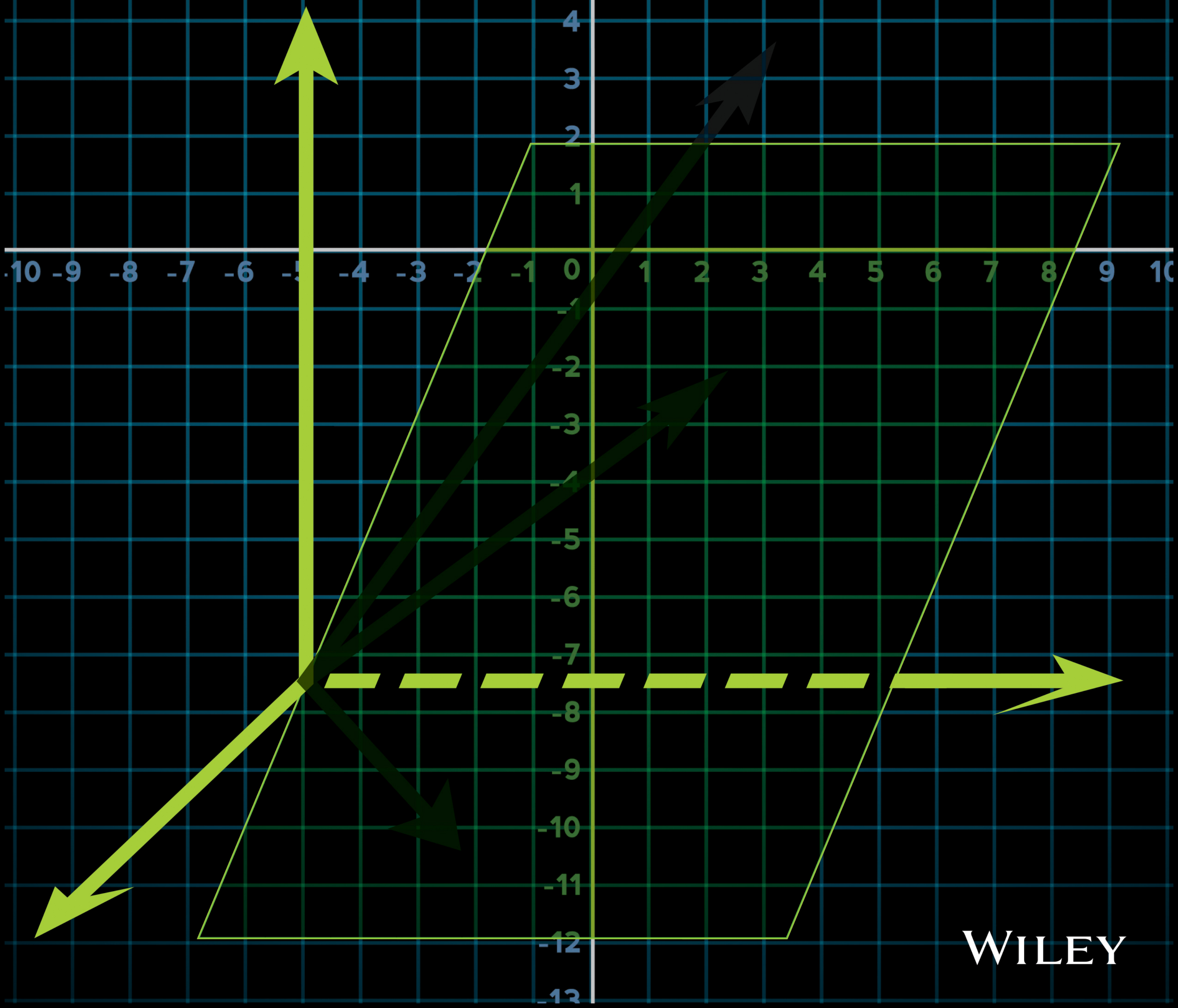
ELEMENTARY LINEAR ALGEBRA

HOWARD ANTON • ANTON KAUL

TWELFTH EDITION



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Elementary Linear Algebra

12th Edition

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WILEY

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10 9 8 7 6 5 4 3 2 1

To
My wife, Pat

My children, Brian, David, and Lauren

My parents, Shirley and Benjamin

In memory of Prof. Leon Bahar,
who fostered my love of mathematics

My benefactor, Stephen Girard (1750–1831),
whose philanthropy changed my life

Howard Anton

To
My wife, Michelle, and my boys, Ulysses and Seth

Anton Kaul

About the Authors

HOWARD ANTON obtained his B.A. from Lehigh University, his M.A. from the University of Illinois, and his Ph.D. from the Polytechnic Institute of Brooklyn (now part of New York University), all in mathematics. In the early 1960s he was employed by the Burroughs Corporation at Cape Canaveral, Florida, where he worked on mathematical problems in the manned space program. In 1968 he joined the Mathematics Department of Drexel University, where he taught and did research until 1983. Since then he has devoted the majority of his time to textbook writing and activities for mathematical associations. Dr. Anton was president of the Eastern Pennsylvania and Delaware Section of the Mathematical Association of America, served on the Board of Governors of that organization, and guided the creation of its Student Chapters. He is the coauthor of a popular calculus text and has authored numerous research papers in functional analysis, topology, and approximation theory. His textbooks are among the most widely used in the world. There are now more than 200 versions of his books, including translations into Spanish, Arabic, Portuguese, Italian, Indonesian, French, and Japanese. For relaxation, Dr. Anton enjoys travel and photography. This text is the recipient of the *Textbook Excellence Award* by Textbook & Academic Authors Association.

ANTON KAUL received his B.S. from UC Davis and his M.S. and Ph.D. from Oregon State University. He held positions at the University of South Florida and Tufts University before joining the faculty at Cal Poly, San Luis Obispo in 2003, where he is currently a professor in the Mathematics Department. In addition to his work on mathematics textbooks, Dr. Kaul has done research in the area of geometric group theory and has published journal articles on Coxeter groups and their automorphisms. He is also an avid baseball fan and old-time banjo player.

Preface

We are proud that this book is the recipient of the *Textbook Excellence Award* from the Text & Academic Authors Association. Its quality owes much to the many professors who have taken the time to write and share their pedagogical expertise. We thank them all.

This 12th edition of *Elementary Linear Algebra* has a new contemporary design, many new exercises, and some organizational changes suggested by the classroom experience of many users. However, the fundamental philosophy of this book has not changed. It provides an introductory treatment of linear algebra that is suitable for a first undergraduate course. Its aim is to present the fundamentals of the subject in the clearest possible way, with sound pedagogy being the main consideration. Although calculus is not a prerequisite, some optional material here is clearly marked for students with a calculus background. If desired, that material can be omitted without loss of continuity. Technology is not required to use this text. However, clearly marked exercises that require technology are included for those who would like to use MATLAB, Mathematica, Maple, or other software with linear algebra capabilities. Supporting data files are posted on both of the following sites:

www.howardanton.com
www.wiley.com/college/anton

Summary of Changes in this Edition

Many parts of the text have been revised based on an extensive set of reviews. Here are the primary changes:

- **Earlier Linear Transformations** — Selected material on linear transformations that was covered later in the previous edition has been moved to Chapter 1 to provide a more complete early introduction to the topic. Specifically, some of the material in Sections 4.10 and 4.11 of the previous edition was extracted to form the new Section 1.9, and the remaining material is now in Section 8.6.
- **New Section 4.3 Devoted to Spanning Sets** — Section 4.2 of the previous edition dealt with both subspaces and spanning sets. Classroom experience has suggested that too many concepts were being introduced at once, so we have slowed down the pace and split off the material on spanning sets to create a new Section 4.3.
- **New Examples** — New examples have been added, where needed, to support the exercise sets.
- **New Exercises** — New exercises have been added with special attention to the expanded early introduction to linear transformations.

Alternative Version

As detailed on the front endpapers, this version of the text includes numerous real-world applications. However, instructors who want to cover a range of applications in more detail might consider the alternative version of this text, *Elementary Linear Algebra with Applications* by Howard Anton, Chris Rorres, and Anton Kaul (ISBN 978-1-119-40672-3). That version contains the first nine chapters of this text plus a tenth chapter with 20 detailed applications. Additional applications, listed in the Table of Contents, can be found on the the websites that accompany this text.

Hallmark Features

- **Interrelationships Among Concepts** — One of our main pedagogical goals is to convey to the student that linear algebra is not a collection of isolated definitions and techniques, but is rather a cohesive subject with interrelated ideas. One way in which we do this is by using a crescendo of theorems labeled "Equivalent Statements" that continually revisit relationships among systems of equations, matrices, determinants, vectors, linear transformations, and eigenvalues. To get a general sense of this pedagogical technique see Theorems 1.5.3, 1.6.4, 2.3.8, 4.9.8, 5.1.5, 6.4.5, and 8.2.4.
- **Smooth Transition to Abstraction** — Because the transition from Euclidean spaces to general vector spaces is difficult for many students, considerable effort is devoted to explaining the purpose of abstraction and helping the student to "visualize" abstract ideas by drawing analogies to familiar geometric ideas.
- **Mathematical Precision** — We try to be as mathematically precise as is reasonable for students at this level. But we recognize that mathematical precision is something to be learned, so proofs are presented in a patient style that is tailored for beginners.
- **Suitability for a Diverse Audience** — The text is designed to serve the needs of students in engineering, computer science, biology, physics, business, and economics, as well as those majoring in mathematics.
- **Historical Notes** — We feel that it is important to give students a sense of mathematical history and to convey that real people created the mathematical theorems and equations they are studying. Accordingly, we have included numerous "Historical Notes" that put various topics in historical perspective.

About the Exercises

- **Graded Exercise Sets** — Each exercise set begins with routine drill problems and progresses to problems with more substance. These are followed by three categories of problems, the first focusing on proofs, the second on true/false exercises, and the third on problems requiring technology. This compartmentalization is designed to simplify the instructor's task of selecting exercises for homework.
- **True/False Exercises** — The true/false exercises are designed to check conceptual understanding and logical reasoning. To avoid pure guesswork, the students are required to justify their responses in some way.
- **Proof Exercises** — Linear algebra courses vary widely in their emphasis on proofs, so exercises involving proofs have been grouped for easy identification. Appendix A provides students some guidance on proving theorems.
- **Technology Exercises** — Exercises that require technology have also been grouped. To avoid burdening the student with typing, the relevant data files have been posted on the websites that accompany this text.
- **Supplementary Exercises** — Each chapter ends with a set of exercises that draws from all the sections in the chapter.

Supplementary Materials for Students Available on the Web

- **Self Testing Review** — This edition also has an exciting new supplement, called the *Linear Algebra Flash-Card Review*. It is a self-study testing system based on the SQ3R study method that students can use to check their mastery of virtually every fundamental concept in this text. It is integrated into WileyPlus, and is available as a free app for iPads. The app can be obtained from the Apple Store by searching for:

  Anton Linear Algebra FlashCard Review

- **Student Solutions Manual** — This supplement provides detailed solutions to most odd-numbered exercises.
- **Maple Data Files** — Data files in Maple format for the technology exercises that are posted on the websites that accompany this text.
- **Mathematica Data Files** — Data files in Mathematica format for the technology exercises that are posted on the websites that accompany this text.
- **MATLAB Data Files** — Data files in MATLAB format for the technology exercises that are posted on the websites that accompany this text.
- **CSV Data Files** — Data files in CSV format for the technology exercises that are posted on the websites that accompany this text.
- **How to Read and Do Proofs** — A series of videos created by Prof. Daniel Solow of the Weatherhead School of Management, Case Western Reserve University, that present various strategies for proving theorems. These are available through WileyPLUS as well as the websites that accompany this text. There is also a guide for locating the appropriate videos for specific proofs in the text.
- **MATLAB Linear Algebra Manual and Laboratory Projects** — This supplement contains a set of laboratory projects written by Prof. Dan Seth of West Texas A&M University. It is designed to help students learn key linear algebra concepts by using MATLAB and is available in PDF form without charge to students at schools adopting the 12th edition of this text.
- **Data Files** — The data files needed for the MATLAB Linear Algebra Manual and Lab Projects supplement.
- **How to Open and Use MATLAB Files** — Instructional document on how to download, open, and use the MATLAB files accompanying this text.

Supplementary Materials for Instructors

- **Instructor Solutions Manual** — This supplement provides worked-out solutions to most exercises in the text.
- **PowerPoint Slides** — A series of slides that display important definitions, examples, graphics, and theorems in the book. These can also be distributed to students as review materials or to simplify note-taking.
- **Test Bank** — Test questions and sample examinations in PDF or LaTeX form.
- **Image Gallery** — Digital repository of images from the text that instructors may use to generate their own PowerPoint slides.
- **WileyPLUS** — An online environment for effective teaching and learning. WileyPLUS builds student confidence by taking the guesswork out of studying and by providing a clear roadmap of what to do, how to do it, and whether it was done right. Its purpose is to motivate and foster initiative so instructors can have a greater impact on classroom achievement and beyond.
- **WileyPLUS Question Index** — This document lists every question in the current WileyPLUS course and provides the name, associated learning objective, question type, and difficulty level for each. If available, it also shows the correlation between the previous edition WileyPLUS question and the current WileyPLUS question, so instructors can conveniently see the evolution of a question and reuse it from previous semester assignments.

A Guide for the Instructor

Although linear algebra courses vary widely in content and philosophy, most courses fall into two categories, those with roughly 40 lectures, and those with roughly 30 lectures. Accordingly, we have created the following long and short templates as possible starting points for constructing your own course outline. Keep in mind that these are just guides, and we fully expect that you will want to customize them to fit your own interests and requirements. Neither of these sample templates includes applications, so keep that in mind as you work with them.

	Long Template	Short Template
Chapter 1: Systems of Linear Equations and Matrices	8 lectures	6 lectures
Chapter 2: Determinants	3 lectures	3 lectures
Chapter 3: Euclidean Vector Spaces	4 lectures	3 lectures
Chapter 4: General Vector Spaces	8 lectures	7 lectures
Chapter 5: Eigenvalues and Eigenvectors	3 lectures	3 lectures
Chapter 6: Inner Product Spaces	3 lectures	2 lectures
Chapter 7: Diagonalization and Quadratic Forms	4 lectures	3 lectures
Chapter 8: General Linear Transformations	4 lectures	2 lectures
Chapter 9: Numerical Methods	2 lectures	1 lecture
Total:	**39 lectures**	**30 lectures**

Reviewers

The following people reviewed the plans for this edition, critiqued much of the content, and provided insightful pedagogical advice:

Charles Ekene Chika, *University of Texas at Dallas*

Marian Hukle, *University of Kansas*

Bin Jiang, *Portland State University*

Mike Panahi, *El Centro College*

Christopher Rasmussen, *Wesleyan University*

Nathan Reff, *The College at Brockport: SUNY*

Mark Smith, *Miami University*

Rebecca Swanson, *Colorado School of Mines*

R. Scott Williams, *University of Central Oklahoma*

Pablo Zafra, *Kean University*

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Our deep appreciation is due to a number of people who have contributed to this edition in many ways:

Prof. Mark Smith, who critiqued the FlashCard program and suggested valuable improvements to the text exposition.

Prof. Derek Hein, whose keen eye helped us to correct some subtle inaccuracies.

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The Wiley Team, Laurie Rosatone, Terri Ward, Melissa Whelan, Tom Kulesa, Kimberly Eskin, Crystal Franks, Laura Abrams, Billy Ray, and Tom Nery each of whom contributed their experience, skill, and expertise to the project.

HOWARD ANTON
ANTON KAUL

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CHAPTER 1

Systems of Linear Equations and Matrices

CHAPTER CONTENTS

Introduction

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called "matrices" (plural of "matrix"). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution of the system can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs for solving systems of equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a multitude of practical applications. It is the study of matrices and related topics that forms the mathematical field that we call "linear algebra." In this chapter we will begin our study of matrices.

1.1 Introduction to Systems of Linear Equations

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. In this first section we will introduce some basic terminology and discuss a method for solving such systems.

Linear Equations

Recall that in two dimensions a line in a rectangular xy-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a, b \text{ not both } 0)$$

and in three dimensions a plane in a rectangular xyz-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a, b, c \text{ not all } 0)$$

These are examples of "linear equations," the first being a linear equation in the variables x and y and the second a linear equation in the variables x, y, and z. More generally, we define a ***linear equation*** in the n variables $x_1, x_2, \dots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \dots, a_n$ and b are constants, and the a's are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$$a_1x + a_2y = b \tag{2}$$

$$a_1x + a_2y + a_3z = b \tag{3}$$

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \dots, x_n$.

EXAMPLE 1 | Linear Equations

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$

$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns***. For example, system (5) that follows has unknowns x and y, and system (6) has unknowns x_1, x_2, and x_3.

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \tag{5–6}$$

A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{7}$$

The double subscripting on the coefficients a_{ij} of the unknowns gives their location in the system—the first subscript indicates the equation in which the coefficient occurs, and the second indicates which unknown it multiplies. Thus, a_{12} is in the first equation and multiplies x_2.

A ***solution*** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

makes each equation a true statement. For example, the system in (5) has the solution

$$x = 1, \quad y = -2$$

and the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

These solutions can be written more succinctly as

$$(1, -2) \quad \text{and} \quad (1, 2, -1)$$

in which the names of the variables are omitted. This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space. More generally, a solution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

of a linear system in n unknowns can be written as

$$(s_1, s_2, \ldots, s_n)$$

which is called an ***ordered n-tuple***. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then the n-tuple is called an ***ordered pair***, and if $n = 3$, then it is called an ***ordered triple***.

Linear Systems in Two and Three Unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities (**Figure 1.1.1**):

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

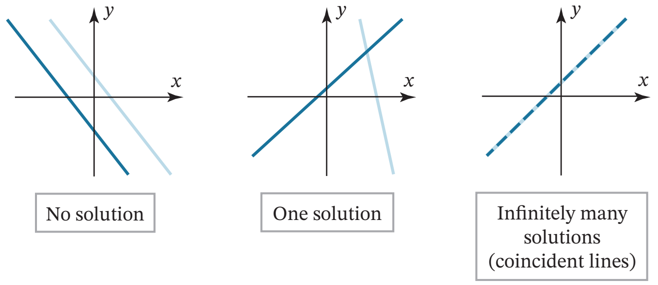

FIGURE 1.1.1

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a *consistent* linear system of two equations in two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions (**Figure 1.1.2**).

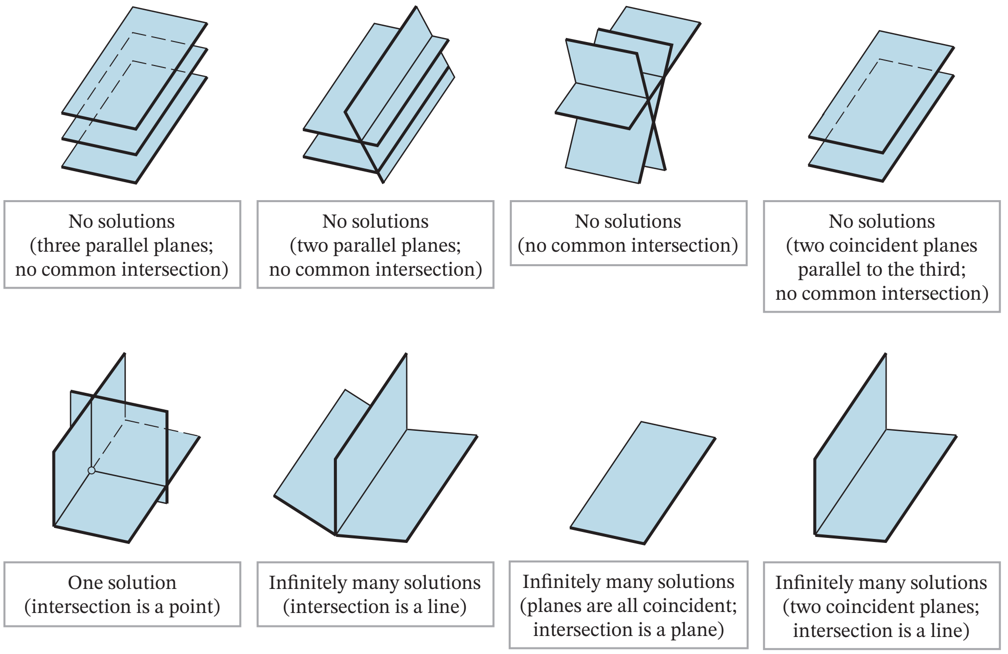

FIGURE 1.1.2

We will prove later that our observations about the number of solutions of linear systems of two equations in two unknowns and linear systems of three equations in three unknowns actually hold for *all* linear systems. That is:

> *Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

EXAMPLE 2 | A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution

$$x = \tfrac{7}{3}, \quad y = \tfrac{4}{3}$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines.

EXAMPLE 3 | A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different y-intercepts.

EXAMPLE 4 | A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and y and hence can be omitted. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \tag{8}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. One way to describe the solution set is to solve this equation for x in terms of y to

In Example 4 we could have also obtained parametric equations for the solutions by solving (8) for y in terms of x and letting $x = t$ be the parameter. The resulting parametric equations would look different but would define the same solution set.

obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value t (called a ***parameter***) to y. This allows us to express the solution by the pair of equations (called ***parametric equations***)

$$x = \tfrac{1}{4} + \tfrac{1}{2}t, \quad y = t$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For example, $t = 0$ yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$ yields the solution $\left(\frac{3}{4}, 1\right)$, and $t = -1$ yields the solution $\left(-\frac{1}{4}, -1\right)$. You can confirm that these are solutions by substituting their coordinates into the given equations.

EXAMPLE 5 | A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - y + 2z = 5 \tag{9}$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of (9). We can do this by first solving this equation for x in terms of y and z, then assigning arbitrary values r and s (parameters) to these two variables, and then expressing the solution by the three parametric equations

$$x = 5 + r - 2s, \quad y = r, \quad z = s$$

Specific solutions can be obtained by choosing numerical values for the parameters r and s. For example, taking $r = 1$ and $s = 0$ yields the solution $(6, 1, 0)$.

Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involved in finding solutions. The required computations can be made more manageable by simplifying notation and standardizing procedures. For example, by mentally keeping track of the location of the +'s, the x's, and the ='s in the linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

As noted in the introduction to this chapter, the term "matrix" is used in mathematics to denote a rectangular array of numbers. In a later section we will study matrices in detail, but for now we will only be concerned with augmented matrices for linear systems.

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Historical Note

Maxime Bôcher (1867–1918)

The first known use of augmented matrices appeared between 200 B.C. and 100 B.C. in a Chinese manuscript entitled *Nine Chapters of Mathematical Art*. The coefficients were arranged in columns rather than in rows, as today, but remarkably the system was solved by performing a succession of operations on the columns. The actual use of the term *augmented matrix* appears to have been introduced by the American mathematician Maxime Bôcher in his book *Introduction to Higher Algebra*, published in 1907. In addition to being an outstanding research mathematician and an expert in Latin, chemistry, philosophy, zoology, geography, meteorology, art, and music, Bôcher was an outstanding expositor of mathematics whose elementary textbooks were greatly appreciated by students and are still in demand today.

[Image: HUP Bocher, Maxime (1), olvwork650836]

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

EXAMPLE 6 | Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by -2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \qquad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad\quad &= 1 \\ y \qquad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1,\ y = 2,\ z = 3$ is now evident.

The solution in this example can also be expressed as the ordered triple $(1, 2, 3)$ with the understanding that the numbers in the triple are in the same order as the variables in the system, namely, x, y, z.

Exercise Set 1.1

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

a. $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ **b.** $x_1 + 3x_2 + x_1x_3 = 2$

c. $x_1 = -7x_2 + 3x_3$ **d.** $x_1^{-2} + x_2 + 8x_3 = 5$

e. $x_1^{3/5} - 2x_2 + x_3 = 4$ **f.** $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

2. In each part, determine whether the equation is linear in x and y.

a. $2^{1/3}x + \sqrt{3}y = 1$ **b.** $2x^{1/3} + 3\sqrt{y} = 1$

c. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ **d.** $\frac{\pi}{7}\cos x - 4y = 0$

e. $xy = 1$ **f.** $y + 7 = x$

3. Using the notation of Formula (7), write down a general linear system of

a. two equations in two unknowns.

b. three equations in three unknowns.

c. two equations in four unknowns.

4. Write down the augmented matrix for each of the linear systems in Exercise 3.

In each part of Exercises 5–6, find a system of linear equations in the unknowns $x_1, x_2, x_3, \ldots$, that corresponds to the given augmented matrix.

5. a. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

6. a. $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

In each part of Exercises 7–8, find the augmented matrix for the linear system.

7. a. $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ **b.** $\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$

c. $\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$

8. a. $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ **b.** $\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

c. $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$

9. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

a. $(3, 1, 1)$ **b.** $(3, -1, 1)$ **c.** $(13, 5, 2)$

d. $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ **e.** $(17, 7, 5)$

10. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} x + 2y - 2z &= 3 \\ 3x - y + z &= 1 \\ -x + 5y - 5z &= 5 \end{aligned}$$

a. $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ **b.** $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$ **c.** $(5, 8, 1)$

d. $\left(\frac{5}{7}, \frac{10}{7}, \frac{2}{7}\right)$ **e.** $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$

11. In each part, solve the linear system, if possible, and use the result to determine whether the lines represented by the equations in the system have zero, one, or infinitely many points of intersection. If there is a single point of intersection, give its coordinates, and if there are infinitely many, find parametric equations for them.

a. $\begin{aligned} 3x - 2y &= 4 \\ 6x - 4y &= 9 \end{aligned}$ **b.** $\begin{aligned} 2x - 4y &= 1 \\ 4x - 8y &= 2 \end{aligned}$ **c.** $\begin{aligned} x - 2y &= 0 \\ x - 4y &= 8 \end{aligned}$

12. Under what conditions on a and b will the linear system have no solutions, one solution, infinitely many solutions?

$$\begin{aligned} 2x - 3y &= a \\ 4x - 6y &= b \end{aligned}$$

In each part of Exercises 13–14, use parametric equations to describe the solution set of the linear equation.

13. a. $7x - 5y = 3$

b. $3x_1 - 5x_2 + 4x_3 = 7$

c. $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

d. $3v - 8w + 2x - y + 4z = 0$

14. a. $x + 10y = 2$

b. $x_1 + 3x_2 - 12x_3 = 3$

c. $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

d. $v + w + x - 5y + 7z = 0$

In Exercises 15–16, each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

15. a. $\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$

b. $\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 3x_1 + 9x_2 - 3x_3 &= -12 \\ -x_1 - 3x_2 + x_3 &= 4 \end{aligned}$

16. a. $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$ **b.** $\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$

In Exercises 17–18, find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

17. a. $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

18. a. $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ **b.** $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

In Exercises 19–20, find all values of k for which the given augmented matrix corresponds to a consistent linear system.

19. a. $\begin{bmatrix} 1 & k & -4 \\ 4 & 8 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & k & -1 \\ 4 & 8 & -4 \end{bmatrix}$

20. a. $\begin{bmatrix} 3 & -4 & k \\ -6 & 8 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} k & 1 & -2 \\ 4 & -1 & 2 \end{bmatrix}$